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Diffusion in a random medium with long-range correlations

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Abstract. A renormalisation group analysis of anomalous diffusion in a random medium with constrained long-range correlations is carried out to two-loop order. The crossover between the long-range correlated model and its short-range correlated counterpart is investigated and the dynamic exponent z is shown to be continuous at the crossover.

1. Introduction

Several authors have recently investigated the effect of random disorder on classical transport processes using a model of a random walk in a stationary random drift field. This model is a generalisation of the usual model of a diffusion process to the case of a system with quenched disorder and it may describe various physical situations, e.g. classical conductivity in the presence of randomly distributed charged impurities [1], diffusion in a fluid with stationary random velocity field [2, 3] or critical dynamics in a system with 'frozen' fluctuations [4]. This 'random random walk' problem is also related to the recent model of a 'true' self-avoiding random walk [5], and it has even been suggested that a model of this type could provide a clue to the solution of the famous $1/f$ problem [6].

The asymptotic properties of diffusion in a medium with short-range correlated random drift have been studied using the renormalisation group (RG) approach both in the case of unconstrained (isotropic) drift [7, 8] and in the case of disorder with constraints (i.e. with independent curl-free and divergence-free parts of the random drift) [2-4, 9]. In order to distinguish these models from their 'long-range' counterparts, which shall be defined below, we shall refer to them as to the short-range case, although actually the constraints imposed on the drift field lead to long-range spatial correlations. Above the upper critical dimension $d > d_c = 2$ normal diffusion is shown to take place in the long-time limit, while at $d \leqslant 2$ the following patterns of anomalous diffusive behaviour occur: purely divergence-free drift leads to superdiffusive behaviour; when both components of the drift are non-zero diffusion remains normal at two dimensions and at $d < 2$ exhibits a subdiffusive anomaly, which does not depend on the relative impact of the components; in the most subtle case of purely curl-free drift vanishing of the RG beta function [9-11] leads to non-universal subdiffusive behaviour at two dimensions and below it to the strong disorder regime, which cannot be treated in the perturbative RG approach.

A generalisation of this model to the case of drift with power-like behaviour of correlations in the momentum space (the 'long-range' model mentioned above) in the case of an unconstrained drift field has been analysed at the one-loop level as a special case of the 'true' self-avoiding random walk problem with long-range interaction [12] and in the generic form with some perturbatively exact results [10, 13]. This analysis has revealed the following properties of the diffusive behaviour in the long-range model. Diffusion is normal above the upper critical dimension $d > d_c = 2+2\alpha$, where α is the exponent of the power-like falloff $1/q^{2\alpha}$ of the disorder field correlations in the momentum space. When $d \leq d_c$, three different anomalous patterns of diffusion may take place. First, in the case of purely transverse drift field the anomalous dimension of time is related to the beta function of the model in such a way that the fixed-point equation determines this anomalous dimension exactly to all orders in perturbation theory [13]. It should be noted that this result is true also in the short-range case. This anomalous dimension corresponds to superdiffusive behaviour. Second, in the generic case (i.e. when both the longitudinal and transverse components of the drift are present) asymptotic behaviour in the model is controlled by a *fixed line* rather than a fixed point of the RG equation. Hence, the anomaly of diffusion is non-universal, depending on the relative strength of transverse and longitudinal parts of the disorder field: the former tends to enhance diffusion, while the latter tends to suppress it. Third, in the case of purely curl-free drift the situation is the same as in the short-range case: the beta function vanishes, which results in non-universal subdiffusive behaviour at $d = 2+2\alpha$ and a non-perturbative regime at $d < 2+2\alpha$. Unfortunate as it is from the point of view of the full RG analysis of the model, this possibly leaves a loophole [10] (through a strong disorder regime) for the logarithmic diffusion of Marinari *et al* [6] to exist at $d < 2+2\alpha$ within the framework of continuum models.

The purpose of this paper is to present a two-loop renormalisation group analysis of anomalous diffusion in a random environment with constrained long-range correlated random drift, including the analysis of the crossover to the case with short-range correlations. The correlation functions of the Gaussian long-range random drift fall off like $1/q^{2\alpha}$ in the momentum space, and formally the short-range case corresponds to $\alpha = 0$. We have calculated the two-loop contributions to beta functions and the dynamic exponent z in the long-range case and found that they contain poles in the parameter α. Thus, not only do the anomalous dimensions of the long-range model differ from those of the short-range model in the limit $\alpha \to 0$, but they even diverge in this limit. These divergences are due to a 'dangerous' irrelevant four-point interaction and we show that, when the effect of this interaction is taken into account, the long-range fixed point becomes unstable near two dimensions resulting in a crossover to a mixed regime, which is characterised by the interplay of both the long-range and the short-range correlations. This mixed regime has been recently analysed by Gevorkian and Lozovik [14] in the case of isotropic drift and we extend their treatment to the generic case. Further, from the mixed regime a crossover to the short-range case may take place and we show at the lowest non-trivial order that the dynamic exponent z does not have discontinuities at the crossovers.

This paper is set up as follows. In § 2 we introduce the field-theoretic treatment of the stochastic problem of a random walk in a random medium. In § 3, renormalisation of the field-theoretic model is carried out and in § 4 the unusual properties of renormalisation group equations and anomalous diffusion are discussed. In § 5 we analyse the crossover between the short-range and the long-range model and show that it does not lead to discontinuities in the dynamic exponent z. Section 6 contains concluding remarks.

2. Field-theoretic approach to diffusion in a random environment

In the continuum limit the problem of a random walk in a random environment is described by the diffusion equation

$$\partial_t \varphi(t, \boldsymbol{x}) = D\Delta\varphi(t, \boldsymbol{x}) - \nabla[\boldsymbol{F}(\boldsymbol{x})\varphi(t, \boldsymbol{x})] \tag{1}$$

where φ is the probability distribution of random walks and $\boldsymbol{F}$ is a Gaussian random field consisting of divergence-free and curl-free parts:

$$\boldsymbol{F} = \boldsymbol{E} + \boldsymbol{B} \qquad \nabla \times \boldsymbol{B} = 0 \qquad \nabla \boldsymbol{E} = 0$$

with zero means and independent variances:

$$\langle \boldsymbol{E} \rangle = \langle \boldsymbol{B} \rangle = \langle \boldsymbol{BE} \rangle = 0$$

$$\langle E_i(\boldsymbol{p}) E_j(\boldsymbol{q}) \rangle = \lambda_{\mathrm{L}}(2\pi)^d \delta(\boldsymbol{p}+\boldsymbol{q}) \frac{q_i q_j}{q^2} \frac{1}{q^{2\alpha}} \tag{2}$$

$$\langle B_i(\boldsymbol{p}) B_j(\boldsymbol{q}) \rangle = \lambda_{\mathrm{T}}(2\pi)^d \delta(\boldsymbol{p}+\boldsymbol{q}) \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) \frac{1}{q^{2\alpha}}$$

where λ_{L} and λ_{T} are, respectively, the coupling constants of longitudinal and transverse components of the drift and d is the dimension of space.

Casting the stochastic problem (1) and (2) into the standard field-theoretic form we arrive at the effective action (we use the same notation for fields and their Fourier transforms):

$$\begin{aligned} S = -\tfrac{1}{2} \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y} \{ \lambda_{\mathrm{L}}^{-1} E_i(\boldsymbol{x}) K_{\parallel ij}^{-1}(\boldsymbol{x}-\boldsymbol{y}) E_j(\boldsymbol{y}) + \lambda_{\mathrm{T}}^{-1} B_i(\boldsymbol{x}) K_{\perp ij}^{-1}(\boldsymbol{x}-\boldsymbol{y}) B_j(\boldsymbol{y}) \} \\ + \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x}\, \tilde{\varphi}(t, \boldsymbol{x}) \{ -\partial_t \varphi(t, \boldsymbol{x}) + D\Delta\varphi(t, \boldsymbol{x}) \\ - \boldsymbol{\nabla}[\boldsymbol{E}(\boldsymbol{x})\varphi(t, \boldsymbol{x})] - \nabla[\boldsymbol{B}(\boldsymbol{x})\varphi(t, \boldsymbol{x})] \} \end{aligned} \tag{3}$$

where the kernels $K_\parallel$ and $K_\perp$ are defined through the Fourier transforms:

$$K_{\parallel jk}(\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} \exp(\mathrm{i}\boldsymbol{x}\boldsymbol{q}) \frac{q_j q_k}{q^2} \frac{1}{q^{2\alpha}}$$

$$K_{\perp jk}(\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} \exp(\mathrm{i}\boldsymbol{x}\boldsymbol{q}) \left(\delta_{jk} - \frac{q_j q_k}{q^2} \right) \frac{1}{q^{2\alpha}}$$

and $\tilde{\varphi}$ is the response field. Correlation and response functions are calculated as functional averages with the weight exp S. As will be shown later, above two dimensions this action is multiplicatively renormalisable as it stands. Therefore we prefer not to integrate out the drift field in order to deal with local interactions instead of the non-local ones, which would appear upon excluding the drift.

We shall characterise diffusion in this system by the mean-square displacement of random walks. As a function of time it is assumed to have a power-like behaviour (up to logarithmic corrections) in the long-time limit:

$$\langle \overline{x^2(t)} \rangle \sim t^{2/z} \tag{4}$$

where z is the dynamic exponent to be calculated in the RG framework. The bar denotes averaging over the distribution φ and the angle brackets denote averaging over the random drift. In terms of the field theory (3), the mean-square displacement of a random walk, which started from the origin $x=0$ at the initial moment of time $t=0$, is related to the Green function of the diffusion equation (1) in the following way:

$$\langle\overline{x^2(t)}\rangle = -\frac{\partial^2}{\partial q_a \partial q_a}\int \mathrm{d}\omega\, \exp(-\mathrm{i}\omega t) G(\omega, \boldsymbol{q})|_{\boldsymbol{q}=0} \tag{5}$$

where G is the Fourier transform of the response function (i.e. the Green function of (1) averaged over the random field) of the model (3):

$$G(\omega, \boldsymbol{q}) = \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x}\, \exp[\mathrm{i}\omega(t-\tilde{t}) - \mathrm{i}\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{y})]\langle\langle\varphi(t, \boldsymbol{x})\tilde{\varphi}(\tilde{t}, \boldsymbol{y})\rangle\rangle. \tag{6}$$

Here double angle brackets denote functional average with the weight exp S.

The diagrammatic representation of the perturbation expansion of correlation and response functions (Green functions of the field theory (3)) is constructed of elements depicted in figure 1, where the directed full curve represents the free propagator of fields φ and $\tilde{\varphi}$ (the arrow points from the $\tilde{\varphi}$ end to the φ end):

$$g_{\varphi\tilde{\varphi}}(\omega, \boldsymbol{q}) = \frac{1}{-\mathrm{i}\omega + Dq^2} \tag{7}$$

and the broken curve corresponds to the sum $D_{jk} = D_{\perp jk} + D_{\parallel jk}$ of the propagators of the transverse vector field $\boldsymbol{B}$ and the longitudinal vector field $\boldsymbol{E}$:

$$D_{jk}(\boldsymbol{q}) = \lambda_{\mathrm{T}}\left(\delta_{jk} - \frac{q_j q_k}{q^2}\right)\frac{1}{q^{2\alpha}} + \lambda_{\mathrm{L}}\frac{q_j q_k}{q^2}\frac{1}{q^{2\alpha}}. \tag{8}$$

The difference between the two interaction terms of (3) is effectively included in the definition (8), and therefore only one interaction vertex (also shown in figure 1) appears in the diagrams. The following properties of this perturbation expansion should be noted. First, as a consequence of stationarity of the drift, there is no frequency flow through the broken curves. Second, due to causality, all graphs containing closed loops of full curves vanish. Since the frequency integrals should be taken over these loops only, in remaining graphs there are no frequency integrations: external frequencies flow freely through full curves. Actually, one may consider them as a natural infrared cutoff in the theory. Third, owing to the invariance of the action (3) with respect to the transformation $\varphi \to s\varphi$, $\tilde{\varphi} \to \tilde{\varphi}/s$ the fields φ and $\tilde{\varphi}$ appear only pairwise in the Green functions of the model.

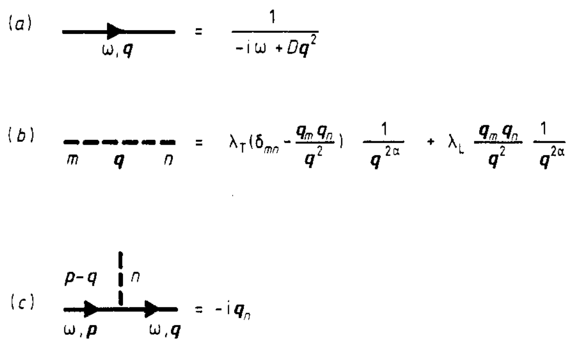

Figure 1. (a) Full line representing the propagator $g_{\varphi\tilde{\varphi}}(\omega, \boldsymbol{q})$; the arrow points from the $\tilde{\varphi}$ end to the φ end. (b) Broken line representing the vector propagator $D_{ij}(\boldsymbol{q})$. (c) Graphical representation of the effective interaction vertex.

For the dimensions of fields and parameters with respect to scaling

$$\boldsymbol{q} \to \mu \boldsymbol{q} \qquad \omega \to \mu^2 \omega \tag{9}$$

we obtain

$$\begin{aligned} & d_\varphi + d_{\tilde\varphi} = d && d_E = d_B = d/2 - \alpha \\ & d_{\lambda_L} = d_{\lambda_T} = 2 + 2\alpha - d && d_D = 0. \end{aligned} \tag{10}$$

The action (3) contains the fields φ and $\tilde\varphi$ in pairs. Therefore only the sum of their dimensions $d_\varphi + d_{\tilde\varphi}$ is determined unambiguously. Coupling constants λ_T and λ_L are dimensionless at the upper critical dimension $d = d_c = 2 + 2\alpha$, and $d_c - d = \varepsilon$ expansions of the anomalous dimensions of parameters and fields will be constructed in the next section using the field-theoretic renormalisation group. It should be noted that, apart from the overall scaling (9), the effective expansion parameters shall be invariant with respect to distinct scale transformations of frequencies and momenta. Therefore, instead of λ_L and λ_T, one should use

$$u \sim \frac{\lambda_L}{D^2} \mu^{-d_{\lambda_L}} \qquad v \sim \frac{\lambda_T}{D^2} \mu^{-d_{\lambda_T}} \tag{11}$$

as the appropriate dimensionless expansion parameters.

3. Renormalisation to two-loop order

In order to establish the renormalisability of the model (3), we proceed with the usual power counting [15]. When we take into account that (i) there are no frequency integrations in the non-vanishing graphs and (ii) the interaction terms in (3) involve the gradient of the response field $\nabla\tilde\varphi$, and the corresponding momenta may be extracted from loop integrals for each external $\tilde\varphi$ line, which reduces the degree of divergence by the number of these lines, we obtain the following expression for the effective degree of divergence δ' of a one-particle-irreducible (1PI) graph with external φ and $\tilde\varphi$ legs:

$$\delta' = 2 + 2\alpha - (n_T + n_L) - n(1 + 2\alpha) \tag{12}$$

where n_T and n_L are numbers of external lines of transverse and longitudinal fields, respectively, and n is the number of *pairs* of external lines of fields φ and $\tilde\varphi$. Graphs without external φ and $\tilde\varphi$ legs do not contain loop integrals and thus for them the degree of divergence is meaningless. The relation (12) implies that for positive α (corresponding to $d_c > 2$) primitive divergencies may occur only in the one-particle-irreducible (1PI) Green functions $\Gamma_{\varphi\tilde\varphi}$, $\Gamma_{\varphi\tilde\varphi E}$ and $\Gamma_{\varphi\tilde\varphi B}$, with $\delta' = 1$, $\delta' = 0$ and $\delta' = 0$, respectively. When $\alpha \leq 0$ (i.e. $d_c \leq 2$) four-point and higher Green functions become relevant and corresponding terms should be added to the action (3). We shall consider the limit $\alpha \to 0$ in § 5, but otherwise the condition $\alpha > 0$ is assumed to hold.

Since the two-point function $\Gamma_{\varphi\tilde\varphi}$ diverges only linearly, there are no divergent contributions to the term $\tilde\varphi\partial_t\varphi$, and the action (3) may be renormalised by the counter-terms

$$\delta S = \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x}\, \tilde\varphi[(Z_D - 1)D\Delta\varphi - (Z_1 - 1)\nabla(\boldsymbol{E}\varphi) - (Z_2 - 1)\nabla(\boldsymbol{B}\varphi)]. \tag{13}$$

Taking into account these counterterms, we obtain the renormalised action in terms of dimensionless coupling constants u and v in the form

$$S=-\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\{C_\alpha(uD^2\mu^\varepsilon)^{-1}E_i(x)K^{-1}_{\parallel ij}(x-y)E_j(y)$$
$$+C_\alpha(vD^2\mu^\varepsilon)^{-1}B_i(x)K^{-1}_{\perp ij}(x-y)B_j(y)\}$$
$$+\int \mathrm{d}t\,\mathrm{d}x\,\tilde{\varphi}(t,x)\{-\partial_t\varphi(t,x)+Z_D D\Delta\varphi(t,x)-Z_1\nabla[\boldsymbol{E}(x)\varphi(t,x)]$$
$$-Z_2\nabla[\boldsymbol{B}(x)\varphi(t,x)]\} \tag{14}$$

where the usual scaling factor μ has been introduced and $\varepsilon=2+2\alpha-d$. The factor C_α, defined by

$$C_\alpha=2\pi^{1+\alpha}/(2\pi)^{2+2\alpha}\Gamma(1+\alpha) \tag{15}$$

where Γ is the gamma function, has been extracted for convenience of calculation. The form of the action (14) implies the following renormalisation of parameters (note that we do not renormalise the fields):

$$D\to D_0=Z_D D \qquad u\to u_0=\mu^\varepsilon Z_1^2 Z_D^{-2}u \qquad v\to v_0=\mu^\varepsilon Z_2^2 Z_D^{-2}v. \tag{16}$$

We shall use dimensional regularisation with minimal subtractions. Then it is not difficult to see that the vertex renormalisation constants Z_1 and Z_2 are equal to all orders in perturbation theory: they are generated by graphs, which differ only in a prefactor at the external line of the vector field, while the loop integrals (from which the renormalisation constants are extracted) are the same for both vertex renormalisations. Thus, we obtain

$$Z_1=Z_2 \tag{17}$$

which is a relation which is responsible for the unusual anomalous behaviour of this model in the generic case of non-zero curl-free and divergence-free parts of the drift field.

In the minimal subtraction scheme, the renormalisation constants are presented in the form of Laurent series in ε with non-vanishing (in the limit $\varepsilon\to 0$) terms only:

$$Z_i=1+\frac{Z_i^{(1)}}{\varepsilon}+\frac{Z_i^{(2)}}{\varepsilon^2}+\dots. \tag{18}$$

Only the residues of simple poles of these expansions are needed to determine the coefficient functions of the RG equations. We have calculated the renormalisation constants Z_D and Z_1 to the two-loop order. The corresponding self-energy and vertex graphs are shown in figures 2 and 3, respectively. For the residue $Z_D^{(1)}$ of the simple pole in the expansion of the renormalisation constant of the diffusion coefficient D we obtain

$$Z_D^{(1)}=\frac{u-(1+2\alpha)v}{2(1+\alpha)}+\frac{[u-(1+2\alpha)v]^2}{16(1+\alpha)^3}+\frac{[-u^2+2(2+\alpha^2)uv+(1+2\alpha)v^2]}{16(1+\alpha)^3} \tag{19}$$

where the first term is the contribution of the one-loop self-energy graph figure 2(a) and the second and third terms are, respectively, the contributions of the two-loop

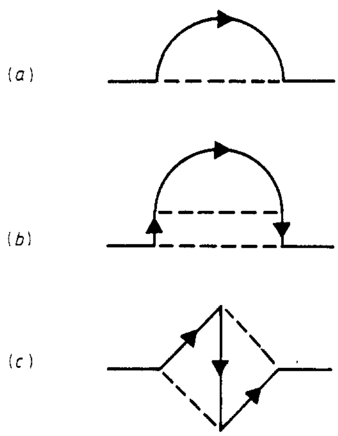

Figure 2. One-loop self-energy graph (*a*) and two-loop self-energy graphs (*b*) and (*c*), which renormalise the diffusion coefficient *D*.

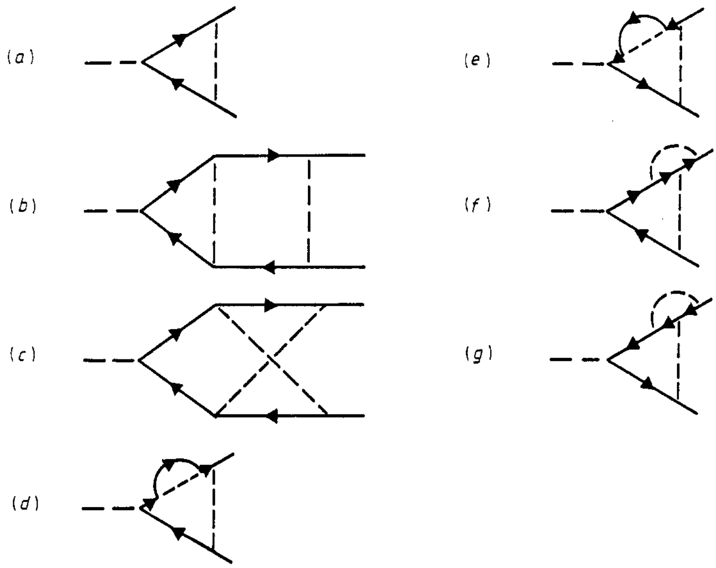

Figure 3. One-loop vertex graph (*a*) and two-loop vertex graphs (*b*)-(*g*), which renormalise the effective interaction. Two-loops graphs (*b*) and (*c*) give rise to divergent in α terms.

graphs figures 2(*b*) and (*c*). The residue term $Z_1^{(1)}$ of the vertex renormalisation constant is of the form

$$Z_1^{(1)} = Z_2^{(1)} = \frac{u}{2(1+\alpha)} - \frac{u[(1-2\alpha^2)u + \alpha(1+\alpha)(1+2\alpha)v]}{16\alpha(1+\alpha)^3} + \frac{u[u+(1+2\alpha)v]}{16\alpha(1+\alpha)^2}$$
$$- \frac{u[(1+2\alpha)u - v]}{16(1+\alpha)^3} - \frac{u[(1+2\alpha)u - v]}{16(1+\alpha)^3} + \frac{(1+2\alpha)uv}{16(1+\alpha)^3} + \frac{(1+2\alpha)u^2}{16(1+\alpha)^3} \tag{20}$$

where the first term is the contribution of the one-loop vertex graph figure 3(*a*) and the following terms are contributions of the two-loop graphs figure 3(*b*)-(*g*), respectively. Note the poles in α, which appear in the contributions of the graphs figures 3(*b*) and (*c*).

4. Renormalisation group equations and anomalous dimensions

Independence of the renormalised two-point Green function G of the arbitrary scale parameter μ is expressed by the equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_u\frac{\partial}{\partial u}+\beta_v\frac{\partial}{\partial v}+\gamma_D D\frac{\partial}{\partial D}\right)G(\omega, \boldsymbol{q})=0 \tag{21}$$

where

$$\beta_{e_i}=\mu\frac{\partial}{\partial\mu}\bigg|_0 e_i=e_i(-\varepsilon+2\gamma_i-2\gamma_D) \qquad e_i=(u, v) \tag{22}$$

and

$$\gamma_x=-\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_x=\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)Z_x^{(1)} \qquad x=(1, 2, D). \tag{23}$$

The partial derivatives in these formulae are taken with fixed values of the bare parameters D_0, u_0 and v_0. Dimensional analysis leads to the relation

$$G(t, \boldsymbol{q}; D, u, v, \mu)=f(t\mu^2 D, \boldsymbol{q}/\mu; u, v) \tag{24}$$

which allows us to rewrite the RG equation (21) in the form

$$\left((2+\gamma_D)t\frac{\partial}{\partial t}-\boldsymbol{q}\frac{\partial}{\partial\boldsymbol{q}}+\beta_u\frac{\partial}{\partial u}+\beta_v\frac{\partial}{\partial v}\right)G(\omega, \boldsymbol{q})=0. \tag{25}$$

For the mean-square displacement we thus obtain

$$\left((2+\gamma_D)t\frac{\partial}{\partial t}+\beta_u\frac{\partial}{\partial u}+\beta_v\frac{\partial}{\partial v}-2\right)\langle\overline{x^2(t)}\rangle=0. \tag{26}$$

Equation (25) defines the dimensions of variables at the RG fixed point. In particular, γ_D gives the anomalous dimension of time. This anomalous dimension is related to the dynamic exponent z of (4) as follows:

$$z=2+\gamma_D(u_*, v_*) \tag{27}$$

where u_* and v_* are the values of coupling constants at the infrared-stable fixed point of RG equations. As usual, the trivial (Gaussian) fixed point $u_*=0$, $v_*=0$ is stable above the upper critical dimension $d>2+2\alpha$, and leads to normal diffusion with linearly growing mean-square displacement: $\langle x^2(t)\rangle\sim t$. When $d\leqslant d_c=2+2\alpha$, renormalisation gives rise to three different patterns of anomalous diffusion, which shall be analysed below.

If the random drift field is transverse, then the vertex renormalisation constant is trivial for both long-range ($\alpha>0$) and short-range ($\alpha=0$) models: $Z_2=1$. This follows from the fact that, in this case integrating by parts, one may extract also the momentum of the external φ line from the graphs of $\Gamma_{\varphi\tilde{\varphi}B}$, thus transforming the formally logarithmically divergent loop integrals to convergent ones. Moreover, for the same reason the four-point Green function $\Gamma_{\varphi\varphi\tilde{\varphi}\tilde{\varphi}}$, which becomes marginal at $\alpha=0$, actually remains finite in the limit $\varepsilon\to0$ and thus our results for the transverse case hold also in the limit $\alpha\to0$ (formally, all 1PI Green functions except $\Gamma_{\varphi\tilde{\varphi}}$ also remain finite for $\alpha<0$, although there is probably no useful meaning in the transverse model for $\alpha\leqslant-\frac{1}{2}$, i.e. $d_c\leqslant1$). We therefore conclude that $\gamma_2=0$ and

$$\beta_v=v(-\varepsilon-2\gamma_D). \tag{28}$$

This relation implies that the fixed-point equation $\beta_v=0$ determines the anomalous dimension γ_D exactly to all orders in perturbation theory, as in the field-theoretic approach to strong turbulence [16, 17]. Nevertheless, γ_D should be calculated in order to find out the character of this fixed point. From (19) and (23) we obtain

$$\gamma_D=-\frac{(1+2\alpha)}{2(1+\alpha)}v+\frac{(1+2\alpha)}{4(1+\alpha)^2}v^2. \tag{29}$$

Equations (28) and (29) confirm to two-loop order the perturbative stability of the non-trivial fixed point $v_*>0$, determined by $\gamma_D(v_*)=\varepsilon/2$. In this case the anomalous behaviour of the mean-square displacement of the random walk is superdiffusive and given by

$$\langle\overline{x^2(t)}\rangle\sim t^{2/(2-\varepsilon/2)} \tag{30}$$

where $\varepsilon=2+2\alpha-d>0$ and $\alpha\geqslant 0$. We emphasise that this expression is exact for both the long-range and the short-range case (it does not even depend on α explicitly). At the upper critical dimension $d=d_c=2+2\alpha$, the behaviour is also superdiffusive with

$$\langle\overline{x^2(t)}\rangle\sim t(\ln t)^{1/2}. \tag{31}$$

In the generic case, when both longitudinal and transverse components of the drift are present, the fact that vertex renormalisation constants Z_1 and Z_2 are equal leads to the remarkable feature that, instead of the usual non-trivial fixed point, the RG equations corresponding to (22) and (23) have a *fixed line.* The fixed line is defined by the equation

$$2\gamma_1(u,v)-2\gamma_D(u,v)-\varepsilon=0 \tag{32}$$

and at two-loop order accuracy is represented by a hyperbola:

$$\frac{(1+2\alpha)}{2(1+\alpha)^2}v^2-\frac{(1+2\alpha)(1+\alpha+\alpha^2)}{4\alpha(1+\alpha)^3}uv-\frac{(1+2\alpha)}{(1+\alpha)}v+\varepsilon=0 \tag{33}$$

which intersects the v axis at the point

$$v=\frac{(1+\alpha)}{(1+2\alpha)}\varepsilon+\frac{(1+\alpha)}{2(1+2\alpha)^2}\varepsilon^2+O(\varepsilon^3)$$

and asymptotically approaches the u axis from the right in the (v, u) plane. Although the detailed form of the fixed line is subject to higher-order corrections, its asymptotic behaviour in the vicinity of the u axis remains the same due to vanishing (see below) on the u axis of loop integral corrections to the longitudinal beta function β_u. From equations (17), (22) and (23) it follows that

$$\mu\frac{\partial}{\partial\mu}\bigg|_0\ln u=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln v. \tag{34}$$

Therefore the ratio of running coupling constants remains fixed in the course of renormalisation:

$$\frac{u(\mu)}{v(\mu)}=\frac{u(1)}{v(1)}=\kappa \tag{35}$$

where $\mu=1$ refers to the initial momentum scale. For calculations in perturbation theory, it is convenient to choose the transverse coupling constant v as the primary coupling constant, the flow of which shall be defined from the corresponding RG equation, while the flow of the longitudinal coupling constant u is determined by the relation (35). To two-loop order, calculations yield

$$\beta_v = v\left(-\varepsilon + \frac{v(1+2\alpha)}{(1+\alpha)} + \frac{v^2(1+2\alpha)}{4\alpha(1+\alpha)^3}[(1+\alpha+\alpha^2)\kappa - 2\alpha(1+\alpha)]\right) \tag{36}$$

where κ is the constant ratio of coupling constants (35). The expression (36) shows that the fixed line (33) is perturbatively infrared stable and therefore controls the long-distance and long-time behaviour of this model. As a consequence of the asymptotic behaviour of the fixed line, however, for large enough ratios $\kappa = u(1)/v(1)$ the relevant part of the fixed line obviously lies beyond the region of applicability of the perturbation theory. The renormalisation group flow of coupling constants u and v corresponding to equations (32)-(35) is sketched in figure 4.

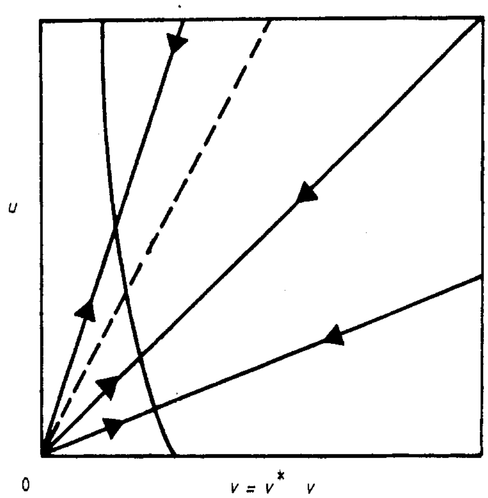

Figure 4. Schematic RG flow of renormalised coupling constants in the (v, u) plane. The arrows show the direction of RG flow with decreasing momentum scale and the broken line corresponds to the borderline case, when $\kappa = \kappa_0 = 1 + 2\alpha - (1-\alpha)\varepsilon + O(\varepsilon^2)$. Above this line the model exhibits subdiffusive behaviour and superdiffusive below it.

The anomalous behaviour, controlled by perturbative RG, is not universal. In particular, the dynamic exponent z at the fixed line takes the form

$$z = 2 + \frac{\varepsilon}{2}\left(\frac{\kappa}{1+2\alpha} - 1\right) + \varepsilon^2\kappa\frac{[1+7\alpha+3\alpha^2-2\alpha^3-\kappa(1+\alpha+\alpha^2)]}{8\alpha(1+\alpha)(1+2\alpha)^2} \tag{37}$$

with explicit dependence on the relative strength (through κ) of longitudinal and transverse parts of disorder. At the upper critical dimension $d = d_c = 2 + 2\alpha$, $\alpha > 0$, we obtain logarithmic corrections of the form

$$\langle\overline{x^2(t)}\rangle \sim t(\ln t)^{\frac{1}{2}[1-\kappa/(1+2\alpha)]}. \tag{38}$$

In this case the fixed line coincides with the u axis, and for finite values of κ RG flow drives the system to the endpoint of the fixed line (i.e. to the Gaussian fixed point): $u_* = v_* = 0$. However, as a reminiscence of the non-universal behaviour at $\varepsilon > 0$, the power of the logarithmic anomaly depends on the direction in which this fixed point is being approached in the (v, u) plane.

When the ratio $\kappa = 0$, expressions (37) and (38) reduce to the transverse ones (30) and (31). When κ grows, the anomaly in diffusion changes from superdiffusive $(0 \leq \kappa < \kappa_0)$ to subdiffusive $(\kappa > \kappa_0)$, where the borderline value of $\kappa = \kappa_0$, corresponding to normal diffusion, to two-loop order is equal to $\kappa_0 = 1 + 2\alpha - \varepsilon(1-\alpha) + O(\varepsilon^2)$. At the upper critical dimension $d = d_c = 2 + 2\alpha$ slow-transient corrections similar to those in the mixed short-range case [4, 8] occur:

$$\overline{\langle x^2(t) \rangle} \sim t\left(1 + \frac{(1-\alpha)}{(1+2\alpha)\ln t}\right). \tag{39}$$

For isotropic drift (i.e. for $\kappa = 1$, which in our case, when $\alpha > 0$, corresponds to superdiffusive behaviour) we recover the one-loop result of [10, 12]. However, in the generic case the renormalisation of the four-point function $\Gamma_{\varphi\varphi\tilde{\varphi}\tilde{\varphi}}$ in two dimensions is not trivial, and thus in the limit $\alpha \to 0$ our results do not coincide with those of the short-range model [2, 4, 5]. Moreover, as can be seen from (37), the dynamic exponent z diverges in the limit $\alpha \to 0$. This situation may be cured by taking into account the four-point function $\Gamma_{\varphi\varphi\hat{\varphi}\hat{\varphi}}$, which is irrelevant for $\alpha > 0$, but becomes marginal in the limit $\alpha \to 0$, and leads to divergences in α. We shall deal with this in the next section.

In the case of purely longitudinal drift we have explicitly checked that to two-loop order the loop integral contributions to the β_u function vanish and it thus remains trivial: $\beta_u = -\varepsilon u/2$. This may be shown to be the case to all orders in the perturbation theory. Physical arguments in favour of this conjecture were given by Kravtsov *et al* [9] and a formal proof using these ideas has been given in [11]. A different approach to this problem, based on the geometrical properties of a non-linear σ model, has been suggested in [10]. Triviality of the longitudinal beta function also implies that the contribution of the relevant (for $d \leq 2$) four-point function $\Gamma_{\varphi\varphi\tilde{\varphi}\tilde{\varphi}}$ vanishes in the two-dimensional short-range case. Therefore in the limit $\alpha \to 0$ the long-range results coincide with the short-range ones. Hence, the long-time behaviour of the mean-square displacement of a random walk in a medium with potential drift is controlled by strong disorder effects below the upper critical dimension and, at the upper critical dimension, is non-universal with continuously varying anomalous dimension, for which a two-loop calculation yields

$$z = 2 + \frac{u}{2(1+\alpha)} + O(u^3). \tag{40}$$

Note that the two-loop contribution to the anomalous dimension vanishes, as in the short-range case [3]. It has been argued that, due to the triviality of the longitudinal beta function, all the higher-order contributions to the dynamic exponent vanish [3, 10]. Although this conjecture is supported by two-loop calculations, we do not find the undetailed argument of [3, 10] entirely convincing, and thus consider this an open problem.

5. Crossover between long-range and short-range models

From equations (36) and (37) it can be seen that both the beta functions and the dynamic exponent z diverge in the limit $\alpha \to 0$, and thus this formal short-range limit is singular in the general case of constrained drift field. These extra divergences are due to the four-point subgraphs shown in figure 5 of the vertex graphs figures 3(*b*) and (*c*). These subgraphs are finite when $\alpha > 0$ but they become logarithmically

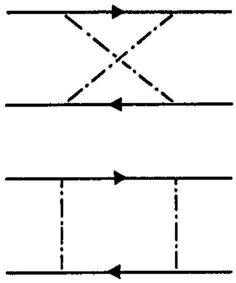

Figure 5. The four-point subgraphs of the two-loop vertex graphs figures 3(*b*) and (*c*), which give rise to the extra divergences in α due to their logarithmic behaviour in two dimensions.

divergent in two dimensions (i.e. when $\alpha = 0$). This expresses the fact that the corresponding four-point interaction, which is irrelevant for $\alpha > 0$, becomes marginal in the limit $\alpha \to 0$. The effect of this 'dangerous irrelevant operator' is seen already at small non-zero values of $\alpha = O(\varepsilon)$, and more exactly the crossover value of α may be found in the standard way [15, 18] by examining the stability of the long-range fixed point against the perturbation by the following composite field operator:

$$O_4 = \int \mathrm{d}t \int \mathrm{d}t' \int \mathrm{d}\boldsymbol{x}\, \varphi(t, \boldsymbol{x}) \nabla \tilde{\varphi}(t, \boldsymbol{x}) \varphi(t', \boldsymbol{x}) \nabla \tilde{\varphi}(t', \boldsymbol{x}). \tag{41}$$

Relevance of this operator is determined by the asymptotic behaviour at small momenta of 1PI Green functions with one insertion of O_4, relative to the behaviour of the same Green functions without the insertion. The asymptotics of Green functions with such operator insertions may be found from the renormalisation of these composite field operators [15, 18]. In our case, there are no composite operators of the same or lower naive dimension as O_4 obeying the symmetries of the model. Therefore the renormalisation may be carried out without operator mixing. For simplicity, we shall consider the four-point 1PI Green function with one O_4 insertion:

$$\Gamma_{4O_4} = \Gamma_{\varphi\varphi\tilde{\varphi}\tilde{\varphi}O_4}(\{\boldsymbol{q}_i\}, \{\omega_j\}; u, v, D, \mu) \tag{42}$$

and its counterpart without the insertion

$$\Gamma_4 = \Gamma_{\varphi\varphi\tilde{\varphi}\tilde{\varphi}}(\{\boldsymbol{q}_i\}, \{\omega_j\}; u, v, D, \mu). \tag{43}$$

We have not renormalised the fields $\tilde{\varphi}$ and φ. Therefore the renormalised (Γ) and bare ($\Gamma^{(0)}$) Green functions are related as follows:

$$\begin{aligned} \Gamma_{4O_4}(u, v, D, \mu) &= Z_{O_4}(u, v) \Gamma^{(0)}_{4O_4}(u_0, v_0, D_0) \\ \Gamma_4(u, v, D, \mu) &= \Gamma^{(0)}_4(u_0, v_0, D_0) \end{aligned} \tag{44}$$

leading to the renormalisation group equations:

$$\begin{aligned} \left(\mu \frac{\partial}{\partial \mu} + \beta_u \frac{\partial}{\partial u} + \beta_v \frac{\partial}{\partial v} + \gamma_D D \frac{\partial}{\partial D} + \gamma_{O_4}\right) \Gamma_{4O_4} &= 0 \\ \left(\mu \frac{\partial}{\partial \mu} + \beta_u \frac{\partial}{\partial u} + \beta_v \frac{\partial}{\partial v} + \gamma_D D \frac{\partial}{\partial D}\right) \Gamma_4 &= 0 \end{aligned} \tag{45}$$

with

$$\gamma_{O_4} = -\mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z_{O_4}. \tag{46}$$

Dimensional analysis yields

$$\begin{aligned}\Gamma_{4O_4}(\{\boldsymbol{q}_i\}, \{\omega_j\}; u, v, D, \mu) &= \mu^2 f_1\left(\left\{\frac{\boldsymbol{q}_j}{\mu}\right\}, \left\{\frac{\omega_j}{\mu^2 D}\right\}; u, v\right)\\ \Gamma_4(\{\boldsymbol{q}_i\}, \{\omega_j\}; u, v, D, \mu) &= D^2\mu^{4-d} f_2\left(\left\{\frac{\boldsymbol{q}_j}{\mu}\right\}, \left\{\frac{\omega_j}{\mu^2 D}\right\}; u, v\right).\end{aligned} \tag{47}$$

From (45) and (47) we obtain

$$\begin{aligned}&\left((2+\gamma_D)\sum_{j=1}^{2}\omega_j\frac{\partial}{\partial\omega_j}+\sum_{i=1}^{3}\boldsymbol{q}_i\frac{\partial}{\partial\boldsymbol{q}_i}-\beta_u\frac{\partial}{\partial u}-\beta_v\frac{\partial}{\partial v}-4+d-2\gamma_D\right)\Gamma_4=0\\ &\left((2+\gamma_D)\sum_{j=1}^{2}\omega_j\frac{\partial}{\partial\omega_j}+\sum_{i=1}^{3}\boldsymbol{q}_i\frac{\partial}{\partial\boldsymbol{q}_i}-\beta_u\frac{\partial}{\partial u}-\beta_v\frac{\partial}{\partial v}-2-\gamma_{O_4}\right)\Gamma_{4O_4}=0.\end{aligned} \tag{48}$$

These equations define the scaling dimensions of the Green functions

$$d_{\Gamma_{4O_4}} = 2+\gamma O_4(u_*, v_*) \qquad d_{\Gamma_4} = 4-d+2\gamma_D(u_*, v_*).$$

By definition, the scaling dimension of the composite operator O_4 is the difference of these dimensions:

$$d_{O_4} = d-2+\gamma_{O_4}(u_*, v_*)-2\gamma_D(u_*, v_*). \tag{49}$$

If this scaling dimension is positive, then the operator O_4 is irrelevant at small momenta and frequencies, otherwise it is marginal or relevant. The anomalous dimension γ_{O_4} is given, at the one-loop order, by the graphs of figure 6 linear in the four-point

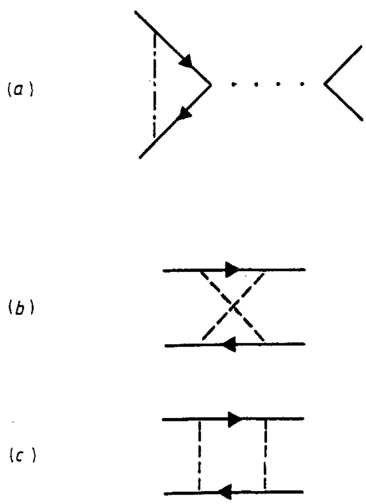

Figure 6. One-loop graphs, which renormalise the four-point interaction vertex. The chain line denotes the sum of the drift field propagator (broken line of figure 1) and the isotropic ($\propto \delta_{mm}$) four-point vertex denoted by the dotted line. Note that there is no frequency flow through either of these lines. For renormalisation of the four-point composite operator O_4 only graphs linear in four-point coupling are needed, while renormalisation of the four-point interaction vertex in the mixed regime includes all these graphs.

interaction given by (41), which we have denoted by the dotted line in order to remind us that there is no frequency flow through the vertex in this direction. Since we are interested in small $\alpha = O(\varepsilon)$, we set $\alpha = 0$ in the one-loop expressions and obtain

$$\gamma_{O_4} = \frac{u-v}{2}. \tag{50}$$

Thus, at one-loop order,

$$d_{O_4} = 2\alpha - \tfrac{1}{2}\varepsilon(1+\kappa) \tag{51}$$

and the operator O_4 becomes marginal (or relevant) when

$$\alpha \leqslant \tfrac{1}{4}\varepsilon(1+\kappa). \tag{52}$$

When this condition is satisfied, the long-range fixed point becomes unstable and the four-point interaction (41) has to be included in the renormalisation scheme.

In this case, the four-point interaction has to be treated on an equal footing with the three-point interaction terms of the action (14). This can be done following Weinrib and Halperin [19], and this analysis has been carried out by Gevorkian and Lozovik [14] for the case of isotropic ($\kappa = 1$) drift. We shall present here a similar treatment of the generic case ($\kappa \neq 1$) with independent longitudinal and transverse couplings. However, it should be noted that the crossover from the pure long-range case to the 'mixed-coupling' regime with competing three-point and four-point interactions cannot be investigated in this scheme, since the four-point term is always present due to the very construction of the renormalisation procedure. On the other hand, a description of the crossover from the mixed-coupling regime to the purely short-range case cannot be included to the composite operator renormalisation treatment presented above. Thus, the full analysis of the crossover between long-range and short-range models should include both methods. On adding the four-point interaction term the renormalised action may be written in the form

$$\begin{aligned} S = -\tfrac{1}{2}\int & \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}[C_\alpha(uD^2\mu^\varepsilon)^{-1}E_i(\boldsymbol{x})K^{-1}_{\parallel ij}(\boldsymbol{x}-\boldsymbol{y})E_j(\boldsymbol{y}) \\ & + C_\alpha(vD^2\mu^\varepsilon)^{-1}B_i(\boldsymbol{x})K^{-1}_{\perp ij}(\boldsymbol{x}-\boldsymbol{y})B_j(\boldsymbol{y})] \\ & + \int \mathrm{d}t\,\mathrm{d}\boldsymbol{x}\,\tilde{\varphi}(t,\boldsymbol{x})\{-\partial_t\varphi(t,\boldsymbol{x}) + Z_D D\Delta\varphi(t,\boldsymbol{x}) \\ & - Z_1\nabla[\boldsymbol{E}(\boldsymbol{x})\varphi(t,\boldsymbol{x})] - Z_2\nabla[\boldsymbol{B}(\boldsymbol{x})\varphi(t,\boldsymbol{x})]\} \\ & + \tfrac{1}{2}C_\alpha^{-1}wD^2\mu^{-\delta}Z_4\int\mathrm{d}t\int\mathrm{d}t'\int\mathrm{d}\boldsymbol{x}\,\varphi(t,\boldsymbol{x})\nabla\tilde{\varphi}(t,\boldsymbol{x})\varphi(t',\boldsymbol{x})\nabla\tilde{\varphi}(t',\boldsymbol{x}) \end{aligned} \tag{53}$$

where w is the totally dimensionless coupling constant, Z_4 is the renormalisation constant of the four-point interaction, C_α is the normalisation factor (15) and the parameter δ is defined as

$$\delta = 2\alpha - \varepsilon = d - 2. \tag{54}$$

The action (53) leads to the following renormalisation of the four-point coupling:

$$w \to w_0 = w\mu^{-\delta}Z_4Z_D^{-2} \tag{55}$$

while the other parameters are renormalised according to (16). From the preceding analysis of the stability of the long-range model it follows that the mixed-coupling regime sets in at $\alpha = O(\varepsilon)$. Therefore, we are interested in the case when $\delta = O(\varepsilon)$, in which the RG analysis is very similar to that of the long-range case. The main difference is that now the divergences of the diagrammatic expansion show in the form of (multiple) poles in the quantities $\Delta = r\delta + p\varepsilon$, where p and r are rational numbers, instead of the usual poles in ε. The expression (18) for the renormalisation constants is thus replaced by the expansion

$$Z_i = 1 + [Z_i]^{(1)} + [Z_i]^{(2)} + \dots \qquad (56)$$

where $[Z_i]^{(1)}$ denotes terms with simple poles in Δ, $[Z_i]^{(2)}$ denotes terms with double poles, etc. The ratio of the three-point coupling constants remains renormalisation invariant. Therefore the behaviour of the system under renormalisation is effectively determined by two beta functions. We use

$$\begin{aligned} \beta_v &= v[-\varepsilon + 2\gamma_2(v, w) - 2\gamma_D(v, w)] \\ \beta_w &= w[\delta + \gamma_4(v, w) - 2\gamma_D(v, w)] \end{aligned} \qquad (57)$$

where

$$\gamma_x = -\mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z_x = \left(\varepsilon v \frac{\partial}{\partial v} - \delta w \frac{\partial}{\partial w}\right)[Z_x]^{(1)} \qquad x = (2, 4, D).$$

At one-loop order we obtain

$$\begin{aligned} \beta_v &= v(-\varepsilon + v + w) \\ \beta_w &= \delta w + \tfrac{1}{2} w^2 + \tfrac{1}{2}(1-\kappa) v w - \tfrac{1}{2}\kappa v^2. \end{aligned} \qquad (58)$$

The one-loop graphs, which contribute to the renormalisation constant Z_4, are shown in figure 6. The four-point vertex contributions to Z_D and Z_2 are given by the graphs figures 2(a) and 3(a), respectively, in which the internal broken line is replaced by the dotted line of figure 6. The RG equations corresponding to (58) have the Gaussian fixed point $v_* = u_* = w_* = 0$, a non-trivial 'short-range' fixed point

$$v_* = u_* = 0 \qquad w_* = -2\delta \qquad (59)$$

and a 'mixed-regime' fixed line

$$v_* = \frac{\varepsilon(\varepsilon + 2\delta)}{2\delta + \varepsilon(1+\kappa)} \qquad u_* = \kappa v_* \qquad w_* = \frac{\kappa \varepsilon^2}{2\delta + \varepsilon(1+\kappa)} \qquad (60)$$

whose stability conditions divide the (δ, ε) plane into three stability regions. In certain parts of the stability region of the mixed-regime fixed line, oscillating corrections to scaling laws may occur (a detailed analysis of this feature in the isotropic case was carried out by Gevorkian and Lozovik [14]). However, this does not affect the values of anomalous dimensions, so therefore we will not dwell on it here. Rather we shall investigate the behaviour of the dynamic exponent z (or, equivalently, the anomalous dimension γ_D) as a function of α. The stability regions of the fixed points as well as the fixed lines (including the long-range fixed line) are depicted in figure 7. Our notation for δ and ε is different from that of Gevorkian and Lozovik; if we label their parameters by the subscript 'GL', then the relation between the parameters is $\varepsilon = \delta_{GL}$ and $\delta = -\varepsilon_{GL}$.

At one-loop order the anomalous dimension of the diffusion coefficient D for the mixed-coupling model (53) is given by

$$\gamma_D(v_*, w_*) = \tfrac{1}{2}(\kappa - 1) v_* \qquad (61)$$

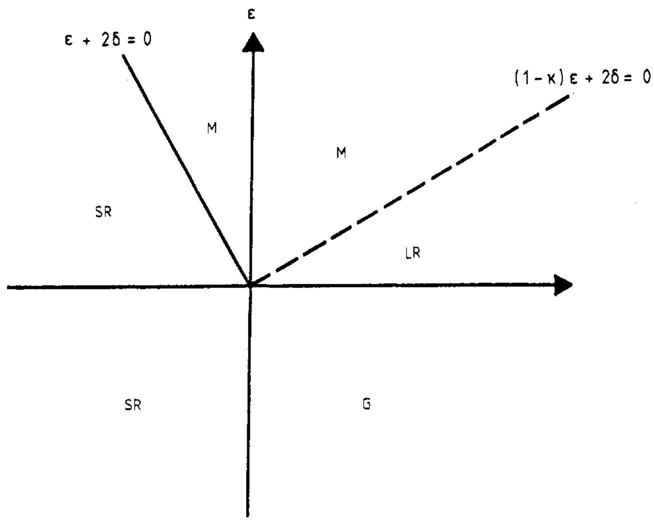

Figure 7. Regions of stability of the RG fixed points and lines in the (δ, ε) plane. The stability region of the Gaussian fixed point is denoted by 'G', short-range fixed point by 'SR', mixed fixed line by 'M' and long-range fixed line by 'LR'. Depending on the value of κ, the border between mixed and long-range regions may lie anywhere between the positive δ axis and the border of mixed and short-range regions.

which, for both the Gaussian and short-range fixed points, yields zero and for the mixed fixed line leads to the expression

$$z = 2 + \frac{\kappa - 1}{2} \frac{\varepsilon(\varepsilon + 2\delta)}{2\delta + \varepsilon(1+\kappa)}. \tag{62}$$

The long-range expression (37) and the mixed-case expression (62) for z do not coincide at the borderline value of α (52). However, it should be taken into account that the long-range formula was obtained for arbitrary finite $\alpha > 0$ and small ε, while the mixed-case expression was derived assuming that $\alpha = O(\varepsilon)$. Expanding the right-hand side of (62) in ε/α, we obtain

$$z = 2 + \frac{\kappa - 1}{2}\varepsilon - \frac{\kappa(\kappa-1)}{8}\frac{\varepsilon^2}{\alpha} + O(\varepsilon^3 \alpha^{-2})$$

which coincides with the expansion in $\alpha = O(\varepsilon)$ of the long-range expression (37) for z. Thus, to this accuracy the long-range and mixed-case expressions for the dynamic exponent z coincide, but this may well be the case to all orders. This connection is obtained for $\alpha = O(\varepsilon)$ and it is not directly related to the borderline value of α (52), which follows from the composite-operator treatment. Since the purely long-range model becomes unstable with respect to the short-range correlations at this value of α, it seems, however, reasonable to regard this as the borderline value of α between the two descriptions. It is also possible that a similar mechanism renders the anomalous dimension continuous in the closely related model of the 'true' self-avoiding random walk [5, 12], but we have not investigated this in detail. At the borderline $\varepsilon + 2\delta = 0$ between the mixed and short-range regimes the one-loop contribution to the mixed-case dynamic exponent (62) vanishes and thus this exponent is continuous to the leading non-trivial order also at this border, since the one-loop contribution to the exponent in the short-range case vanishes identically, as is seen from (59) and (61), and has been earlier shown [2, 4] for $\varepsilon=0$ and [14] for $\varepsilon\neq 0$, $\kappa=1$. Strictly speaking, this applies to the case with $\kappa\neq 1$ only, because the first-order contribution to z is also identically zero in the mixed regime in the case of isotropic ($\kappa=1$) drift [14]. In this case the leading contribution to the anomalous dimension γ_D for the mixed coupling model is given by

$$\gamma_D(v,w)=\frac{w^2}{2}+\frac{(\varepsilon-\delta)vw}{2\varepsilon}-\frac{\delta v^2}{2\varepsilon}. \tag{63}$$

This expression differs from that of Gevorkian and Lozovik [14] due to the different normalisation conditions used. At this mixed fixed line we obtain from (60) and (63)

$$z=2+\tfrac{1}{4}\varepsilon^2-\tfrac{1}{2}\varepsilon\delta \tag{64}$$

which is in agreement with the $\alpha=O(\varepsilon)$ expansion of the long-range expression for z (37) when $\kappa=1$, and at the borderline $\varepsilon=-2\delta$ coincides with the short-range expression [2, 8, 14], $z=2+2\delta^2$, thus confirming the continuity of z to leading non-trivial order also in the case $\kappa=1$.

6. Conclusion

In this paper we have carried out a renormalisation group analysis of the long-time behaviour of diffusion in a random environment with correlations falling off like $1/q^{2\alpha}$ in the momentum space ($1/|\boldsymbol{x}-\boldsymbol{x}'|^{d-2\alpha}$ in the coordinate space). We have calculated the renormalisation group beta functions and the dynamic critical exponent z in the weak disorder limit to two-loop order and found singularities in the form of poles in α. These singularities appear due to a 'dangerous irrelevant' interaction, which has to be included in the renormalisation procedure for small α (i.e. near two dimensions). We have investigated the effect of this interaction at one-loop order and shown that the dynamic exponent z remains a finite and continuous function of the parameter α in the 'short-range' limit $\alpha\to 0$.

For finite $\alpha>0$ below the upper critical dimension $d_c=2+2\alpha$, the long-time behaviour is controlled by an infrared-stable fixed line of the renormalisation group equations and heavily depends on the relative strength of the curl-free and divergence-free parts of the random drift. For purely divergence-free drift the exponent of the corresponding superdiffusive behaviour is determined exactly by the renormalisation group equations. In the presence of both components of the drift field the ε expansion of the dynamic exponent cannot be determined exactly, and to two-loop order it corresponds to superdiffusive behaviour for small enough values of the ratio κ of the longitudinal and transverse coupling constants: $0<\kappa<\kappa_0=1+2\alpha-(1-\alpha)\varepsilon+O(\varepsilon^2)$, leads to normal diffusion at the borderline value $\kappa=\kappa_0$, and subdiffusive behaviour above it. In the limit of vanishing transverse coupling the long-time behaviour of this system is controlled by strong disorder effects and thus remains uncertain in the present perturbative treatment. At the upper critical dimension in the generic case logarithmic corrections to normal diffusion occur, leading to superdiffusive behaviour for $0\leqslant\kappa<\kappa_0$ and subdiffusive for $\kappa>\kappa_0$. In the case of purely longitudinal drift, a power-like subdiffusive anomaly takes place with an exponent continuously depending on the coupling constant.

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